Int J Theor Phys (2007) 46: 3072–3087
DOI 10.1007/s10773-007-9421-3

Symmetries and Motions in NUT-Taub Spinning Space

Akhtara Banu · M. Abdullah Ansary

Received: 23 February 2007 / Accepted: 9 May 2007 /
Published online: 30 May 2007

Abstract We review the geodesic motion of pseudo-classical spinning particles in curved space. We describe the generalized Killing equations for spinning spaces and express the constants of motion. We apply the formalism to solve for the motion of a pseudo-classical Dirac fermion in NUT-Taub spinning space and analyze the motion on a cone and on a plane.

Keywords Spinning space · Geodesics · Symmetries · Killing–Yano tensors

1 Introduction

Spinning particles, such as Dirac fermions, can be described by pseudo-classical mechanics model involving anti-commuting Grassmann variables ψ^μ for the spin degrees freedom [1–9]. The configuration space of the theory, called spinning space, is an extension of a (pseudo-)Euclidean manifold, described by local (real) coordinates $\{x^\mu\}$, to a graded manifold described by local graded coordinates $\{x^\mu, \psi^\mu\}$. Geodesic flow along time-like curves of such a graded manifold with Minkowskian signature $(+---)$ describes the classical limit of the motion of a relativistic point-like particle, which carries a spin $s = \hbar/2$ in quantum mechanics.

Generalizations of Riemannian geometry based on anti-commuting variables have been proved to be of wide mathematical interest. For example, supersymmetric point particle mechanics have found applications in the area of index theorem, while the BRST methods are widely used in the study of topological invariants. For these reasons the study of the geometry of graded pseudo-manifold with the coordinates $\{x^\mu, \psi^\mu\}$ is well motivated.

In this paper we investigate the motion of a pseudo-classical spin-$\frac{1}{2}$ particle in the NUT-Taub space which is a stationary and axisymmetric solution of Einstein's empty space equation. The similarity between the group structure for the NUT symmetry and the group structure for spherical symmetry leads to the interpretation that the NUT-Taub metric corresponds

A. Banu (✉) · M.A. Ansary
Department of Mathematics, Rajshahi University, Rajshahi 6205, Bangladesh
e-mail: akhtara_banu@yahoo.com

to a vacuum cosmological-like solution with periodic time. According to this identification, the NUT-Taub space admits peculiar properties [10, 11]: it does not admit an interpretation without a periodic time coordinate, has no reasonable space-like surface, and is an asymptotically zero curvature space which apparently does not admit asymptotically rectangular coordinates.

The linearized Einstein equations for the NUT-Taub metric are analogous to the case in electromagnetism of a semiinfinite magnetic solenoid or a magnetic monopole. This means, the NUT-Taub metric is a particle-like solution whose spherically symmetric source has both ordinary mass and "magnetic-like" mass.

The NUT-Taub space has the metric [12]

$$ds^2 = -U(dt - 2n\cos\theta d\varphi)^2 + \frac{1}{U}dr^2 + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

where

$$U(r) = 1 - \frac{2}{r^2 + n^2}(Mr + n^2). \tag{2}$$

M is the ordinary mass of the source and n is the NUT parameter, has the identification of the gravitational "magnetic" mass or magnetic monopole [12–16]. In the limit that $n = 0$ the NUT-Taub metric reduces to the Schwarzschild metric.

For a nonzero n the vanishing of dt.$g_{\mu\nu}$ in (1) identifies the singularities at $\theta = 0$ and $\theta = \pi$. Because of this axial singularity the metric admits different physical interpretations. Misner [10, 11] introduced a periodic time coordinate to remove the singularity; but this makes the metric an uninteresting particle-like solution. To avoid a periodic time coordinate, Bonnor [17] removed the singularity at $\theta = 0$ and related the singularity at $\theta = \pi$ to a semiinfinite massless source of angular momentum along the axis of symmetry. This is analogous to representing the magnetic monopole in electromagnetic theory by semiinfinite solenoid [18]. The singularity along z-axis is analogous to the Dirac string.

The NUT-Taub metric possesses properties similar to both the Schwarzschild and the Kerr metrics. Like the Kerr and Schwarzschild, the NUT-Taub space is Petrovtype D and has a Killing horizon at $r_0 = M + \sqrt{M^2 + n^2}$. Like the Schwarzschild metric, the single nonvanishing Riemann curvature scalar is spherically symmetric. Also the NUT-Taub space, like the Schwarzschild space, admits a four parameter group of motion with three space-like generators having the same commutator algebra as do the generators for angular momentum.

The NUT-Taub metric (1) is Kerr-like in regard to that it has a crossed spacetime metric component $g_{t\varphi}$ which generates gravimagnetic effects. In the Kerr metric the cross term breaks spherical symmetry and produces an ergosphere and frame dragging. On the contrary, the cross term in (1) does not generate ergosphere, but it does produce an effect similar to the dragging of inertial frames. Moreover, although the cross term in (1) singles out the z-axis and appears to break spherical symmetry, the space components of the geodesics as a function of proper time are spherically symmetric. However, the geodesic coordinate time component is not spherically symmetric. Since the time component is dependent on the orientation of the "Dirac string", we say that the geodesics are only "almost" spherically symmetric. This suggests that the energy of the "Dirac string" makes contribution to the solution.

The NUT-Taub space, as was suggested by McGuire and Ruffini [19], admits no direct physical interpretation. It is sometimes considered as unphysical. Our study of pseudo-classical spin-$\frac{1}{2}$ particles in such a peculiar space is interesting.

The organization of this paper is as follows. In Sect. 2 we give a brief account of the relevant equations for the motion of spinning particles in curved spaces. The generalized Killing equations for spinning spaces are analyzed and the constants of motion are derived in terms of the solutions of these equations. In Sect. 3 we investigate the motion of pseudo-classical spinning particles in the NUT-Taub space. We examine the generalized Killing equations for this spinning space and derive the constants of motion in terms of the Killing–Yano tensors. In Sect. 4 we solve the equations derived in the previous section for special case of motion on a cone and on a plane. Finally, in Sect. 5 we present our concluding remarks.

2 Symmetries of Spinning Space

Spinning spaces are graded extensions of ordinary Riemannian manifolds; the additional fermionic dimensions are parametrized by vectorial Grassmann coordinates ψ^μ. This type of extension generates a SUSY in the geometry of the graded manifold, which acts on the coordinates as

$$\delta x^\mu = -\mathrm{i}\epsilon\psi^\mu, \qquad \delta\psi^\mu = \epsilon\dot{x}^\mu, \tag{3}$$

where the dot denotes a derivative with respect to proper time and the infinitesimal parameter ϵ of the transformation is Grassmann-odd.

The geodesic for the spinning space can be obtained from the action,

$$S = \int_1^2 d\tau\left(\frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{\mathrm{i}}{2}g_{\mu\nu}(x)\psi^\mu\frac{D\psi^\nu}{D\tau}\right), \tag{4}$$

where the covariant derivative of ψ^μ is defined by

$$\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda\Gamma^\mu_{\lambda\nu}\psi^\nu. \tag{5}$$

The equations of motion of the theory can then be cast in the form

$$\frac{D^2x^\mu}{D\tau^2} = \ddot{x}^\mu + \Gamma^\mu_{\lambda\nu}\dot{x}^\lambda\dot{x}^\nu = -\frac{1}{2}\mathrm{i}\psi^\kappa\psi^\lambda R_{\kappa\lambda}{}^\mu{}_\nu\dot{x}^\nu, \tag{6}$$

$$\frac{D\psi^\mu}{D\tau} = 0. \tag{7}$$

The antisymmetric tensor

$$S^{\mu\nu} = -\mathrm{i}\psi^\mu\psi^\nu \tag{8}$$

can formally be regarded as the spin-polarization tensor of the particle. Equation (6) then implies the existence of a spin-dependent gravitational force

$$\frac{D^2x^\mu}{D\tau^2} = \frac{1}{2}S^{\kappa\lambda}R_{\kappa\lambda}{}^\mu{}_\nu\dot{x}^\nu, \tag{9}$$

while (7) asserts that the spin is covariantly constant:

$$\frac{DS^{\mu\nu}}{D\tau} = 0. \tag{10}$$

The space-like components S^{ij} are proportional to the particle's magnetic dipole moment, while the time-like components S^{i0} represent the electric dipole moment. The electric dipole moment for free Dirac particles vanishes in the rest frame. This gives the covariant constraint [20]

$$g_{\nu\lambda}(x) S^{\mu\nu} \dot{x}^{\lambda} = 0, \tag{11}$$

which in the Grassmann coordinates takes the form

$$g_{\mu\nu}(x) \dot{x}^{\mu} \psi^{\nu} = 0. \tag{12}$$

Let the variations

$$\begin{aligned} \delta x^{\mu} &= \mathcal{R}^{\mu}(x, \dot{x}, \psi) = R^{(1)\mu}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_n} R^{(n+1)\mu}_{\nu_1 \cdots \nu_n}(x, \psi), \\ \delta \psi^{\mu} &= \mathcal{S}^{\mu}(x, \dot{x}, \psi) = S^{(0)\mu}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_n} S^{(n)\mu}_{\nu_1 \cdots \nu_n}(x, \psi) \end{aligned} \tag{13}$$

leave the action (4) invariant modulo boundary terms. The Lagrangian transforms into a total derivative

$$\delta S = \int_1^2 d\tau \frac{d}{d\tau} \left(\delta x^{\mu} p_{\mu} - \frac{\mathrm{i}}{2} \delta \psi^{\mu} g_{\mu\nu} \psi^{\nu} - \mathcal{J}(x, \dot{x}, \psi) \right), \tag{14}$$

where

$$p_{\mu} = \Pi_{\mu} + \frac{\mathrm{i}}{2} \Gamma_{\mu\nu\lambda} \psi^{\lambda} \psi^{\nu}, \quad \Pi_{\mu} = g_{\mu\nu} \dot{x}^{\nu}, \tag{15}$$

p_{μ} is the canonical momentum conjugate to x^{μ}, and Π_{μ} the covariant momentum. If the equations of motion are satisfied, it follows from Noether's theorem that the quantity $\mathcal{J}$ is a constant of motion.

For any constant of motion $\mathcal{J}(x, \pi, \psi)$ the bracket with the world-line Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} \Pi_{\mu} \Pi_{\nu} \tag{16}$$

vanishes:

$$\{H, \mathcal{J}\} = 0 \tag{17}$$

for the Poisson–Dirac brackets defined by

$$\{F, G\} = \mathcal{D}_{\mu} F \frac{\partial G}{\partial \Pi_{\mu}} - \frac{\partial F}{\partial \Pi_{\mu}} \mathcal{D}_{\mu} G - \mathcal{R}_{\mu\nu} \frac{\partial F}{\partial \Pi_{\mu}} \frac{\partial G}{\partial \Pi_{\nu}} + \mathrm{i}(-1)^{a_F} \frac{\partial F}{\partial \psi^{\mu}} \frac{\partial G}{\partial \psi^{\mu}}, \tag{18}$$

where

$$\mathcal{D}_{\mu} = \partial_{\mu} + \Gamma^{\lambda}_{\mu\nu} \Pi_{\lambda} \frac{\partial}{\partial \Pi_{\nu}} - \Gamma^{\lambda}_{\mu\nu} \psi^{\nu} \frac{\partial}{\partial \psi^{\lambda}}, \qquad \mathcal{R}_{\mu\nu} = \frac{\mathrm{i}}{2} \psi^{\kappa} \psi^{\lambda} R_{\kappa\lambda\mu\nu}, \tag{19}$$

and a_F is the Grassmann parity of F: $a_F = (0, 1)$ for F =(even, odd).

With the expansion

$$\mathcal{J} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{J}^{(n)\mu_1 \cdots \mu_n}(x, \psi) \Pi_{\mu_1} \cdots \Pi_{\mu_n}. \tag{20}$$

Equation (17) yields the generalized Killing equations [7, 9]

$$D_{(\mu_{n+1}}\mathcal{J}^{(n)}_{\mu_1\cdots\mu_n)}+\frac{\partial\mathcal{J}^{(n)}_{(\mu_1\cdots\mu_n}}{\partial\psi^\kappa}\Gamma_{\mu_{n+1})\lambda}{}^{\kappa}\psi^\lambda=\frac{\mathrm{i}}{2}\psi^\kappa\psi^\lambda R_{\kappa\lambda\nu(\mu_{n+1}}\mathcal{J}^{(n+1)}_{\mu_1\cdots\mu_n)}{}^{\nu}, \tag{21}$$

where the parentheses denote full symmetrization with norm one over the indices enclosed.

In general, spinning spaces can have two types of symmetries; firstly, there are four independent *generic* symmetries which exist in any theory:

(i) Proper-time translations generated by the Hamiltonian H (16);
(ii) SUSY generated by the supercharge

$$Q=\Pi_\mu\psi^\mu; \tag{22}$$

(iii) Chiral symmetry generated by the chiral charge

$$\Gamma_*=\frac{\mathrm{i}^{[d/2]}}{d!}\sqrt{g}\epsilon_{\mu_1\cdots\mu_d}\psi^{\mu_1}\cdots\psi^{\mu_d}; \tag{23}$$

(iv) Dual SUSY generated by the dual supercharge

$$Q^*=\mathrm{i}\{\Gamma_*,Q\}=\frac{\mathrm{i}^{[d/2]}}{(d-1)!}\sqrt{g}\epsilon_{\mu_1\cdots\mu_d}\Pi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_d} \tag{24}$$

where d is the dimension of the space.

The condition for the absence of an intrinsic electric dipole moment of physical fermions (leptons and quarks) as formulated in (12) gives

$$Q=0. \tag{25}$$

However, to keep the presentation in the following as general as possible, we shall not fix the value of the supercharge.

The second kind of symmetries, called *nongeneric* SUSYs, depend on the explicit form of the metric $g_{\mu\nu}(x)$ [6, 21] and is generated by the phase-space function

$$Q_{\mathrm{f}}=\mathcal{J}^{(1)\mu}\Pi_\mu+\mathcal{J}^{(0)}, \tag{26}$$

where $\mathcal{J}^{(0,1)}(x,\psi)$ are independent of Π. This charge generates the SUSY transformation

$$\delta x^\mu=-\mathrm{i}\epsilon f^\mu{}_a\psi^a\equiv-\mathrm{i}\epsilon\mathcal{J}^{(1)\mu}. \tag{27}$$

Greek and Latin indices refer to world and Lorentz indices, respectively, and are converted into each other by the vielbein $e_\mu{}^a(x)$ and its inverse $e^\mu{}_a(x)$. Substitution of (26) into the generalized Killing equations (21) gives

$$\mathcal{J}^{(0)}(x,\psi)=\frac{\mathrm{i}}{3!}c_{abc}(x)\psi^a\psi^b\psi^c, \tag{28}$$

where

$$c_{abc}=-2D_{[a}f_{bc]},\qquad D_\mu f_{\nu a}+D_\nu f_{\mu a}=0, \tag{29}$$

the square brackets denote full antisymmetrization with norm one over the indices enclosed. If there exists N such symmetries specified by N sets of tensors $(f^{\mu}_{i\,a}, c_{iabc})$, $i = 1, \ldots, N$, the corresponding generators will be

$$Q_i = f^{\mu}_{i\,a} \Pi_\mu \psi^a + \frac{\mathrm{i}}{3!} c_{iabc}(x) \psi^a \psi^b \psi^c. \tag{30}$$

Obviously, for $f^{\mu}{}_a = e^{\mu}{}_a$ and $c_{abc} = 0$, the supercharge (22) is precisely of this form. Hence, the choice $i = 0$: $Q = Q_0$, $e^{\mu}{}_a = f^{\mu}_{0\,a}$, etc., gives the quantities defining the standard SUSY.

The conserved charges Q_i give the algebra

$$\{Q_i, Q_j\} = -2\mathrm{i} Z_{ij}, \tag{31}$$

where

$$Z_{ij} = \frac{1}{2} K^{\mu\nu}_{ij} \Pi_\mu \Pi_\nu + I^{\mu}_{ij} \Pi_\mu + G_{ij}, \tag{32}$$

and

$$K^{\mu\nu}_{ij} = \frac{1}{2}(f^{\mu}_{i\,a} f^{\nu a}_{j} + f^{\nu}_{i\,a} f^{\mu a}_{j}), \tag{33}$$

$$\begin{aligned} I^{\mu}_{ij} &= \frac{1}{2} \mathrm{i} \psi^a \psi^b I^{\mu}_{ijab} \\ &= \frac{1}{2} \mathrm{i} \psi^a \psi^b \left(f^{\nu}_{i\,b} D_\nu f^{\mu}_{j\,a} + f^{\nu}_{j\,b} D_\nu f^{\mu}_{i\,a} + \frac{1}{2} f^{\mu c}_{i} c_{jabc} + \frac{1}{2} f^{\mu c}_{j} c_{iabc} \right), \end{aligned} \tag{34}$$

$$\begin{aligned} G_{ij} &= -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d G_{ijabcd} \\ &= -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d \left(R_{\mu\nu ab} f^{\mu}_{i\,c} f^{\nu}_{j\,d} + \frac{1}{2} c_{iab}{}^{e} c_{jcde} \right). \end{aligned} \tag{35}$$

The $K_{ij\mu\nu}$ is a symmetric Killing tensor of second rank:

$$D_{(\lambda} K_{ij\mu\nu)} = 0, \tag{36}$$

while I^{μ}_{ij} is the corresponding Killing vector:

$$\mathcal{D}_{(\mu} I_{ij\nu)} = \frac{1}{2} \mathrm{i} \psi^a \psi^b D_{(\mu} I_{ij\nu)ab} = \frac{1}{2} \mathrm{i} \psi^a \psi^b R_{ab\lambda(\mu} K_{ij\nu)}{}^{\lambda}, \tag{37}$$

and G_{ij} the corresponding Killing scalar:

$$\mathcal{D}_\mu G_{ij} = -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d D_\mu G_{ijabcd} = \frac{1}{2} \mathrm{i} \psi^a \psi^b R_{ab\lambda\mu} I^{\lambda}_{ij}. \tag{38}$$

The functions Z_{ij} satisfy the generalized Killing equations. Hence their brackets with the Hamiltonian vanish and they are constants of motion, $dZ_{ij}/d\tau = 0$. For $i = j = 0$, (31) reduces to the usual SUSY algebra

$$\{Q, Q\} = -2\mathrm{i} H. \tag{39}$$

When i or j is not equal to zero, Z_{ij} correspond to new bosonic symmetries, unless $K_{ij}^{\mu\nu} = \lambda_{(ij)} g^{\mu\nu}$, with $\lambda_{(ij)}$ a constant (may be zero). Then the corresponding Killing vector I_{ij}^{μ} and scalar G_{ij} disappear identically. Further, the supercharges for $\lambda_{(ij)} \neq 0$ close on the Hamiltonian and hence there exists a second SUSY of the standard type. Thus the theory admits an N-extended SUSY with $N \geq 2$. Again, if there exists a second independent Killing tensor $K^{\mu\nu}$ not proportional to $g^{\mu\nu}$, there exists a genuine new type of SUSY.

The quantity Q_i is a superinvariant

$$\{Q_i, Q\} = 0 \tag{40}$$

for the bracket (18), if and only if

$$K_{0i}^{\mu\nu} = f^{\mu}{}_{a} e^{\nu a} + f^{\nu}{}_{a} e^{\mu a} = 0. \tag{41}$$

Then the full constants of motion Z_{ij} can be constructed directly by repeated differentiation of $f^{\mu}{}_{a}$ [6].

Since Z_{ij} are symmetric in (ij) we can diagonalize them and obtain the algebra

$$\{Q_i, Q_j\} = -2\mathrm{i}\delta_{ij} Z_i, \tag{42}$$

where Z_i are $N+1$ conserved bosonic charges of which the first one is the Hamiltonian: $Z_0 = H$.

3 Motion in NUT-Taub Spinning Space

The NUT-Taub space, described by the metric (1) has an isometry group $SU(2) \times U(1)$. The four Killing vectors are

$$D^{(\alpha)} \equiv R^{(\alpha)\mu} \partial_{\mu}, \quad \alpha = 0, \dots, 3, \tag{43}$$

or explicitly

$$\begin{aligned} D^{(0)} &= \frac{\partial}{\partial t}, \qquad D^{(1)} = -\sin\varphi \frac{\partial}{\partial\theta} - \cot\theta \cos\varphi \frac{\partial}{\partial\varphi} + 2n \cot\theta \cos\varphi \frac{\partial}{\partial t}, \\ D^{(2)} &= \cos\varphi \frac{\partial}{\partial\theta} - \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} + 2n \cot\theta \sin\varphi \frac{\partial}{\partial t}, \\ D^{(3)} &= \frac{\partial}{\partial\varphi} + 2n \frac{\partial}{\partial t}. \end{aligned} \tag{44}$$

$D^{(0)}$, which generates the $U(1)$ of t translation, commutes with the other Killing vectors. The remaining three vectors, corresponding to the invariance of the metric (1) under spatial rotations ($\alpha = 1, 2, 3$), obey an $SU(2)$ algebra with

$$[D^{(i)}, D^{(j)}] = -\varepsilon^{ijk} D^{(k)} \quad (i, j, k = 1, 2, 3). \tag{45}$$

This is contrasted with the Schwarzschild space, where the isometry group at spacelike infinity is $SO(3) \times U(1)$. This illustrates the essential topological character of the magnetic mass [22, 23].

These invariances, in purely bosonic case, would correspond to conservation of the so-called "relative electric charge" and the angular momentum [24–29]:

$$q = -U(\dot{t} - 2n\cos\theta\dot{\varphi}), \tag{46}$$

$$\mathbf{j} = \mathbf{r}\times\mathbf{p} + 2nq\frac{\mathbf{r}}{r}. \tag{47}$$

The first generalized Killing equation of (21) shows that for each Killing vector there is an associated Killing scalar, and if we limit ourselves to variations (13) that terminate after the terms linear in $\dot{x}^\mu$, the corresponding constants of motion would be of the form

$$\mathcal{J}^{(\alpha)} = B^{(\alpha)} + m\dot{x}^\mu R_\mu^{(\alpha)}, \tag{48}$$

which asserts that the Killing scalars $B^{(\alpha)}$ contribute to the "relative electric charge" and the total angular momentum.

For the NUT-Taub space, we obtain

$$\begin{aligned}
B^{(0)} &= VS^{tr} + 2nV\cos\theta S^{r\varphi} - nU\sin\theta S^{\theta\varphi},\\
B^{(1)} &= -2nV\cos\varphi\cot\theta(1+\cos\theta)S^{tr}\\
&\quad + \frac{1}{2}nU\cos\varphi\cos\theta S^{t\theta} - \frac{1}{2}nU\sin\varphi\sin\theta S^{t\varphi} - r\sin\varphi S^{r\theta}\\
&\quad - \cos\varphi\cot\theta[(9n^2V + r)\sin^2\theta + 4n^2V\cos\theta(1+\cos\theta)]S^{r\varphi}\\
&\quad + \cos\varphi[(r^2+n^2)\sin^2\theta + 3n^2U(1-2\cos^2\theta) - 2n^2U\cos\theta]S^{\theta\varphi},\\
B^{(2)} &= -\frac{\partial B^{(1)}}{\partial\varphi},\\
B^{(3)} &= -2nV(1-2\cos\theta)S^{tr} - \frac{1}{2}nU\sin\theta S^{t\theta}\\
&\quad + \left[r\sin^2\theta - 7n^2V\cos^2\theta - \frac{3}{2}n^2V(1+4\cos\theta)\right]S^{r\varphi}\\
&\quad + \sin\theta[(r^2+n^2)\cos\theta - 2n^2U(1-3\cos\theta)]S^{\theta\varphi},
\end{aligned} \tag{49}$$

where $2V = dU/dr$. The conserved total angular momentum in the spinning case is given by

$$\mathbf{J} = \mathbf{B} + \mathbf{j}, \qquad J^{(0)} = B^{(0)} - q \tag{50}$$

with $\mathbf{J} = (J^{(1)}, J^{(2)}, J^{(3)})$ and $\mathbf{B} = (B^{(1)}, B^{(2)}, B^{(3)})$. For the components of $\mathbf{J}$, we obtain

$$\begin{aligned}
J^{(1)} &= B^{(1)} - (r^2+n^2)\sin\varphi\dot{\theta} - (r^2+n^2)\cos\theta\sin\theta\cos\varphi\dot{\varphi} + 2nq\sin\theta\cos\varphi,\\
J^{(2)} &= B^{(2)} + (r^2+n^2)\cos\varphi\dot{\theta} - (r^2+n^2)\cos\theta\sin\theta\sin\varphi\dot{\varphi} + 2nq\sin\theta\sin\varphi,\\
J^{(3)} &= B^{(3)} + (r^2+n^2)\sin^2\theta\dot{\varphi} + 2nq\cos\theta.
\end{aligned} \tag{51}$$

We obtain two interesting relations:

$$J^{(1)}\sin\varphi - J^{(2)}\cos\varphi = -rS^{r\theta} - (r^2+n^2)\dot{\theta} - \frac{1}{2}nU\sin\theta S^{t\varphi}, \tag{52}$$

$$\begin{aligned} & J^{(1)} \sin\theta\cos\varphi + J^{(2)}\sin\theta\sin\varphi + J^{(3)}\cos\theta \\ & \quad = -2nJ^{(0)} + 2nV(2\cos\theta + \sin^2\theta)S^{tr} \\ & \qquad - \frac{1}{2}n^2 V\cos\theta[4\cos\theta(\cos\theta+5)+13]S^{r\varphi} \\ & \qquad + \sin\theta[(r^2+n^2) - n^2U\{1+2\cos\theta(3\cos\theta-2)\}]S^{\theta\varphi}. \end{aligned} \tag{53}$$

The four universal conserved charges, described in the previous section, are given by

1. The energy

$$E = \frac{1}{2U}\dot{r}^2 + \frac{1}{2}(r^2+n^2)(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - \frac{1}{2U}q^2; \tag{54}$$

2. The supercharge

$$Q = \frac{1}{U}\dot{r}\psi^r + (r^2+n^2)\dot{\theta}\psi^\theta + q\psi^t - [2n\cos\theta q - (r^2+n^2)\sin^2\theta\dot{\varphi}]\psi^\varphi; \tag{55}$$

3. The chiral charge

$$\Gamma_* = (r^2+n^2)\sin\theta\psi^r\psi^\theta\psi^\varphi\psi^t; \tag{56}$$

4. The dual supercharge

$$Q^* = (r^2+n^2)\sin\theta(\dot{r}\psi^\theta\psi^\varphi\psi^t - \dot{\theta}\psi^r\psi^\varphi\psi^t + \dot{\varphi}\psi^r\psi^\theta\psi^t - \dot{t}\psi^r\psi^\theta\psi^\varphi). \tag{57}$$

Keeping in mind that ψ^μ is covariantly constant as formulated in (7), we obtain

$$\begin{aligned} \dot{\psi}^t &= \left(\frac{2n^2U}{r^2+n^2}\cot\theta\dot{\theta} - \frac{V}{U}\dot{r}\right)\psi^t - \left(\frac{2nr}{r^2+n^2}\cos\theta\dot{\varphi} + \frac{V}{U^2}q\right)\psi^r \\ &\quad - \left[n\cos\theta(\tan\theta + 2\cot\theta)\dot{\varphi} + \frac{2n^2}{r^2+n^2}\cot\theta q\right]\psi^\theta \\ &\quad - n\cos\theta\left[2\left(\frac{r}{r^2+n^2} - \frac{V}{U}\right)\dot{r} - \left(\tan\theta + 2\cot\theta + \frac{4n^2U}{r^2+n^2}\cot\theta\right)\dot{\theta}\right]\psi^\varphi, \\ \dot{\psi}^r &= [rU - (r^2+n^2)V](\dot{\theta}\psi^\theta + \sin^2\theta\dot{\varphi}\psi^\varphi), \\ \dot{\psi}^\theta &= -\frac{r\dot{\theta}}{r^2+n^2}\psi^r - \frac{r\dot{r}}{r^2+n^2}\psi^\theta + \sin\theta\cos\theta\dot{\varphi}\psi^\varphi \\ &\quad + \frac{n\sin\theta}{r^2+n^2}[(q + 2nU\cos\theta\dot{\varphi})\psi^\varphi - U\dot{\varphi}\psi^t] \\ \dot{\psi}^\varphi &= \frac{nU}{r^2+n^2}\operatorname{cosec}\theta\dot{\theta}\psi^t - \frac{r\dot{\varphi}}{r^2+n^2}\psi^r - \left(\cot\theta\dot{\varphi} + \frac{nq}{r^2+n^2}\operatorname{cosec}\theta\right)\psi^\theta \\ &\quad - \left[\frac{r\dot{r}}{r^2+n^2} + \left(1 + \frac{2n^2U}{r^2+n^2}\right)\cot\theta\dot{\theta}\right]\psi^\varphi. \end{aligned} \tag{58}$$

We now turn to *nongeneric* SUSYs generated by the functions Q_i of (30).
The Killing–Yano tensor $f_{\mu\nu}$ for the metric (1) is given by

$$\frac{1}{2}f_{\mu\nu}dx^\mu \wedge dx^\nu = ndr \wedge (dt - 2n, \cos\theta d\varphi) + r\sin\theta d\theta \wedge (r^2+n^2)d\varphi. \tag{59}$$

For the vierbein $e_\mu{}^a(x)$ we have the following expressions:

$$e_\mu{}^0 dx^\mu = -\sqrt{U}(dt - 2n\cos\theta d\varphi), \qquad e_\mu{}^1 dx^\mu = \frac{1}{\sqrt{U}}dr,$$
$$e_\mu{}^2 dx^\mu = \sqrt{(r^2+n^2)}d\theta, \qquad e_\mu{}^3 dx^\mu = \sqrt{(r^2+n^2)}\sin\theta d\varphi. \tag{60}$$

The components of $f_\mu{}^a(x)$ are given by

$$f_\mu{}^0 dx^\mu = \frac{1}{\sqrt{U}}ndr, \qquad f_\mu{}^1 dx^\mu = -n\sqrt{U}(dt - 2n\cos\theta d\varphi),$$
$$f_\mu{}^2 dx^\mu = -r\sqrt{(r^2+n^2)}\sin\theta d\varphi, \qquad f_\mu{}^3 dx^\mu = r\sqrt{(r^2+n^2)}d\theta. \tag{61}$$

From (29) the components of c_{abc} are obtained follows:

$$c_{012} = 0, \qquad c_{013} = 0, \qquad c_{023} = 0, \qquad c_{123} = -2\sqrt{U}. \tag{62}$$

The new SUSY generator Q_{f} given by (30) takes the following form for the NUT-Taub space:

$$\begin{aligned} Q_{\mathrm{f}} &= -r\sqrt{(r^2+n^2)}[2nq\cos\theta - (r^2+n^2)\sin^2\theta\dot{\varphi}]\psi^\theta \\ &\quad + \left[\frac{2n^2\cos\theta}{\sqrt{U}}\dot{r} - r(r^2+n^2)^{3/2}\sin\theta\dot{\theta}\right]\psi^\varphi \\ &\quad - \frac{n}{\sqrt{U}}[\dot{r}\psi^t - q\psi^r] - 2\mathrm{i}\sqrt{U}\psi^r\psi^\theta\psi^\varphi. \end{aligned} \tag{63}$$

The Killing tensor, vector, and scalar are constructed from (33–35) and are given by

$$K_{\mu\nu}\Pi^\mu\Pi^\nu = -\frac{n^2}{U}(\dot{r}^2 - q^2) + r^2(r^2+n^2)(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2), \tag{64}$$

$$I_\mu\Pi^\mu = \sqrt{(r^2+n^2)U}[r\sin\theta S^{r\varphi}\dot{\varphi} + (nS^{t\varphi} + rS^{r\theta})\dot{\theta}], \tag{65}$$

$$G = -\frac{2n}{r^2+n^2}S^{tr}S^{\theta\varphi}. \tag{66}$$

The new conserved charge is then given by

$$Z = \frac{1}{2}K_{\mu\nu}\Pi^\mu\Pi^\nu + I_\mu\Pi^\mu + G. \tag{67}$$

The equations of this section are to be integrated for the trajectories of the particle in terms of the usual coordinates $\{x^\mu\}$ and Grassmann coordinates $\{\psi^\mu\}$. These equations are quite intricate and the general solution is by no means illuminating. We therefore discuss special solutions in the next section for the motion on a cone and on a plane.

4 Special Solutions

We solve the equations derived in the previous section for special kind of motions of the spin-$\frac{1}{2}$ particle in spinning space.

4.1 Motion on a cone

We choose the z-axis along $\mathbf{J}$. Then the motion of the particle may conveniently be described in terms of polar coordinates

$$\mathbf{r} = r\mathbf{e}(\theta, \varphi), \qquad \mathbf{e} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{68}$$

For this choice of axis we have

$$(r^2+n^2)\dot{\theta} = -rS^{r\theta} - \frac{1}{2}nU\sin\theta S^{t\varphi}, \tag{69}$$

$$\begin{aligned}\dot{\varphi} &= \frac{2nq}{(r^2+n^2)\cos\theta} - \frac{2nV(1+\cos\theta)}{(r^2+n^2)\sin^2\theta}S^{tr} + \frac{nU}{2(r^2+n^2)\sin\theta}S^{t\theta}\\ &\quad - \frac{1}{(r^2+n^2)\sin^2\theta}[(9n^2V+r)\sin^2\theta + 4n^2V\cos\theta(1+\cos\theta)]S^{r\varphi}\\ &\quad + \left[\tan\theta + \frac{3n^2U}{(r^2+n^2)\cos\theta\sin\theta} - \frac{2n^2U}{(r^2+n^2)\sin\theta}(3\cos\theta+1)\right]S^{\theta\varphi}.\end{aligned} \tag{70}$$

The equations of motion for the spin components when $\dot{\theta}=0$ are given by

$$\begin{aligned}\dot{S}^{r\theta} &= -\frac{r\dot{r}}{r^2+n^2}S^{r\theta} + \sin\theta\left[\cos\theta\left(1+\frac{2n^2U}{r^2+n^2}\right)\dot{\varphi} + \frac{nq}{r^2+n^2}\right]S^{r\varphi}\\ &\quad - \frac{nU\sin\theta}{r^2+n^2}\dot{\varphi}S^{rt} - [rU-(r^2+n^2)V]\sin^2\theta\dot{\varphi}S^{\theta\varphi},\\ \dot{S}^{r\varphi} &= -\frac{r\dot{r}}{r^2+n^2}S^{r\varphi} - \left(\cot\theta\dot{\varphi} + \frac{nq}{r^2+n^2}\operatorname{cosec}\theta\right)S^{r\theta},\\ \dot{S}^{\theta\varphi} &= -\frac{2r\dot{r}}{r^2+n^2}S^{\theta\varphi} + \frac{r\dot{\varphi}}{r^2+n^2}S^{r\theta} - \frac{nU\sin\theta}{r^2+n^2}\dot{\varphi}S^{t\varphi},\\ \dot{S}^{\theta t} &= -\left(\frac{r}{r^2+n^2}+\frac{V}{U}\right)\dot{r}S^{\theta t} - 2n\cos\theta\left(\frac{r}{r^2+n^2}-\frac{V}{U}\right)\dot{r}S^{\theta\varphi}\\ &\quad + \sin\theta\left[\cos\theta\left(1+\frac{2n^2U}{r^2+n^2}\right)\dot{\varphi} + \frac{nq}{r^2+n^2}\right]S^{\varphi t}\\ &\quad + \left(\frac{2nr}{r^2+n^2}\cos\theta\dot{\varphi} + \frac{V}{U^2}q\right)S^{r\theta},\\ \dot{S}^{rt} &= [rU-(r^2+n^2)V]\sin^2\theta\dot{\varphi}S^{\varphi t} - \frac{V}{U}\dot{r}S^{rt}\\ &\quad - n\cos\theta\left[(\tan\theta+2\cot\theta)\dot{\varphi} + \frac{2nq}{r^2+n^2}\operatorname{cosec}\theta\right]S^{r\theta}\\ &\quad - 2n\cos\theta\left(\frac{r}{r^2+n^2}-\frac{V}{U}\right)\dot{r}S^{r\varphi},\end{aligned} \tag{71}$$

$$\dot{S}^{\varphi t} = -\frac{r\dot{\varphi}}{r^2+n^2} S^{rt} - \left(\cot\theta\dot{\varphi} + \frac{nq}{r^2+n^2}\operatorname{cosec}\theta\right) S^{\theta t}$$
$$- \left(\frac{r}{r^2+n^2} + \frac{V}{U}\right)\dot{r} S^{\varphi t} + \left(\frac{2n\cos\theta}{r^2+n^2} r\dot{\varphi} + \frac{V}{U^2}q\right) S^{r\varphi}$$
$$+ n\cos\theta\left[(\tan\theta + 2\cot\theta)\dot{\varphi} + \frac{2nq}{r^2+n^2}\operatorname{cosec}\theta\right] S^{\theta\varphi}.$$

Since we are looking for solutions with $\dot{\theta} = 0$, we have from (69),

$$S^{r\theta} + \frac{nU\sin\theta}{2r} S^{t\varphi} = 0. \tag{72}$$

This relation implies $\Gamma_* = 0$.

Using (72) we can express $S^{r\theta}$ through $S^{t\varphi}$. The system of equations (71) reduces to a more tractable form

$$\begin{aligned}
&\dot{S}^{r\varphi} + \frac{r\dot{r}}{r^2+n^2} S^{r\varphi} = \frac{3n^2 U}{2r(r^2+n^2)} q S^{t\varphi}, \\
&\dot{S}^{\theta\varphi} + \frac{2r\dot{r}}{r^2+n^2} S^{\theta\varphi} = -\frac{3n^2 U}{(r^2+n^2)^2} q \tan\theta S^{t\varphi}, \\
&\dot{S}^{\theta t} + \left(\frac{r}{r^2+n^2} + \frac{V}{U}\right)\dot{r} S^{\theta t} \\
&\quad = -2n\cos\theta\left(\frac{r}{r^2+n^2} - \frac{V}{U}\right)\dot{r} S^{\theta\varphi} - \sin\theta\left(\frac{nV}{rU} + \frac{3n}{r^2+n^2} + \frac{6n^3 U}{(r^2+n^2)^2}\right) q S^{t\varphi}, \\
&\dot{S}^{rt} + \frac{V}{U}\dot{r} S^{rt} = -\frac{2n\cos\theta}{r^2+n^2}\left[\left(rU - (r^2+n^2)V - \frac{n^2 U}{2r}\right)\tan^2\theta - \frac{3n^2 U}{2r}\right] q S^{t\varphi} \\
&\qquad - 2n\cos\theta\left(\frac{r}{r^2+n^2} - \frac{V}{U}\right)\dot{r} S^{r\varphi}.
\end{aligned} \tag{73}$$

A particular solution may be found, if one chooses $q = 0$, in the form

$$\begin{aligned}
&S^{r\varphi} = \frac{C^{r\varphi}}{\sqrt{r^2+n^2}}, \qquad S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2}, \\
&S^{\theta t} = \frac{C^{\theta t}}{\sqrt{U(r^2+n^2)}} + n\cos\theta\frac{C^{\theta\varphi}}{r^2+n^2}, \\
&S^{rt} = \frac{C^{rt}}{\sqrt{U}} + n\cos\theta\frac{C^{r\varphi}}{\sqrt{r^2+n^2}}, \\
&S^{r\theta} = \frac{C^{r\theta}}{\sqrt{r^2+n^2}}, \qquad S^{t\varphi} = \frac{C^{t\varphi}}{\sqrt{U(r^2+n^2)}},
\end{aligned} \tag{74}$$

where $C^{\mu\nu}$ are Grassmann constants. The case of choice $S^{t\varphi} = 0$ is included into the case $q = 0$.

The constraint $Q=0$ (see (25)) enables one to solve for ψ^t in terms of the spatial components ψ^i. As a result, it gives $\Gamma_*=Q^*=0$. For the spin components we have

$$\begin{aligned}\dot r S^{r\varphi} &= -qUS^{t\varphi}, \qquad qS^{rt}=[2n\cos\theta q-(r^2+n^2)\sin^2\theta\dot\varphi]S^{r\varphi},\\ \dot r S^{r\theta} &= -U[2n\cos\theta q-(r^2+n^2)\sin^2\theta\dot\varphi]S^{\theta\varphi}+qUS^{\theta t}.\end{aligned}\tag{75}$$

The condition $Q=0$ modifies drastically the form of the solutions.

In spite of the complexity of the equations, we have a simple exact solution for the components of the spin-tensor:

$$S^{\theta\varphi}=\frac{C^{\theta\varphi}}{r^2+n^2}.\tag{76}$$

For the equations of motion we obtain

$$\begin{aligned}\dot t &= 2n\cos\theta\dot\varphi-\frac{q}{U},\\ q &= -J^{(0)}+nU\sin\theta\frac{C^{\theta\varphi}}{r^2+n^2},\\ \dot\varphi &= \frac{1}{r^2+n^2}\left[\frac{2nq}{\cos\theta}+\frac{2n^2U(2-\cos\theta-4\cos^2\theta)}{(r^2+n^2)\sin\theta\cos\theta}C^{\theta\varphi}+\frac{\sin\theta}{\cos\theta}C^{\theta\varphi}\right],\\ \dot r &= \{U[2E-(r^2+n^2)\sin^2\theta\dot\varphi^2]+q^2\}^{1/2}.\end{aligned}\tag{77}$$

4.2 Motion on a plane

Planar motion for spinning particles happen only in two kinds of situations: (i) the orbital angular momentum vanishes, or (ii) spin and orbital angular momentum are parallel.

For the plane we consider $\theta=\pi/2$. Then the equations of motion for the spin components become

$$\begin{aligned}\dot S^{r\theta} &= -\frac{r\dot r}{r^2+n^2}S^{r\theta}+\frac{nq}{r^2+n^2}S^{r\varphi}-\frac{nU}{r^2+n^2}\dot\varphi S^{rt}-[rU-(r^2+n^2)V]\dot\varphi S^{\theta\varphi},\\ \dot S^{r\varphi} &= -\frac{r\dot r}{r^2+n^2}S^{r\varphi}-\frac{nq}{r^2+n^2}S^{r\theta},\\ \dot S^{\theta\varphi} &= -\frac{2r\dot r}{r^2+n^2}S^{\theta\varphi}+\frac{r\dot\varphi}{r^2+n^2}S^{r\theta}-\frac{nU}{r^2+n^2}\dot\varphi S^{t\varphi},\\ \dot S^{\theta t} &= -\left(\frac{r}{r^2+n^2}+\frac{V}{U}\right)\dot r S^{\theta t}+\frac{nq}{r^2+n^2}S^{\varphi t}+\frac{V}{U^2}qS^{r\theta},\\ \dot S^{rt} &= [rU-(r^2+n^2)V]\dot\varphi S^{\varphi t}-\frac{V}{U}\dot r S^{rt},\\ \dot S^{\varphi t} &= -\left(\frac{r}{r^2+n^2}+\frac{V}{U}\right)\dot r S^{\varphi t}+\frac{V}{U^2}qS^{r\varphi}-\frac{r\dot\varphi}{r^2+n^2}S^{rt}-\frac{nq}{r^2+n^2}S^{\theta t}.\end{aligned}\tag{78}$$

From (52) and (53) we obtain

$$\begin{aligned}rS^{r\theta} &= -\frac{1}{2}nUS^{t\varphi},\\ q &= -\frac{1}{2n}(r^2+n^2+n^2U)S^{\theta\varphi}.\end{aligned}\tag{79}$$

Case (i). The solution describes a particle moving along a fixed radius, for which $\dot{\varphi}=0$. We obtain

$$
\begin{aligned}
S^{r\varphi} &= \frac{C^{r\varphi}}{\sqrt{r^2+n^2}}, \qquad S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2},\\
S^{\theta t} &= \frac{C^{\theta t}}{\sqrt{U(r^2+n^2)}}, \qquad S^{rt} = \frac{C^{rt}}{\sqrt{U}},\\
S^{\varphi t} &= \frac{C^{\varphi t}}{\sqrt{U(r^2+n^2)}}.
\end{aligned} \tag{80}
$$

The SUSY constraint $Q=0$ gives a nenule spin component

$$
S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2}. \tag{81}
$$

In this case the orbit of the particle is described by

$$
\begin{aligned}
\dot{r} &= \{2UE+q^2\}^{1/2},\\
q &= -\frac{1}{2n}(r^2+n^2+n^2U)\frac{C^{\theta\varphi}}{r^2+n^2},\\
\dot{t} &= \frac{1}{U}\left(J^{(0)}+nU\frac{C^{\theta\varphi}}{r^2+n^2}\right).
\end{aligned} \tag{82}
$$

Case (ii). The concerned motion is for $\dot{\varphi}\neq 0$. From $Q=0$ we obtain

$$
\frac{\dot{r}}{U}S^{r\theta} = -J^{(3)}S^{\theta\varphi}, \qquad \frac{\dot{r}}{U}S^{rt} = -J^{(3)}S^{t\varphi}. \tag{83}
$$

Interestingly, even in this case a spin component is nenule:

$$
S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2}. \tag{84}
$$

For the orbit of the particle we have

$$
\begin{aligned}
\dot{r} &= \{2UE-U(r^2+n^2)\dot{\varphi}^2+q^2\}^{1/2},\\
q &= -\frac{1}{2n}(r^2+n^2+n^2U)\frac{C^{\theta\varphi}}{r^2+n^2},\\
\dot{\varphi} &= \frac{1}{r^2+n^2}\left(J^{(3)}+2n^2U\frac{C^{\theta\varphi}}{r^2+n^2}\right),\\
\dot{t} &= \frac{1}{U}\left(J^{(0)}+nU\frac{C^{\theta\varphi}}{r^2+n^2}\right).
\end{aligned} \tag{85}
$$

5 Concluding Remarks

The spinning particle model is a worldline supersymmetric extension of the theory of a scalar particle. It describes a relativistic particle with spin-$\frac{1}{2}$.

The main concern of our study has been the motion of pseudo-classical spinning particles in the NUT-Taub space. The supersymmetric extension of this space admits fermionic symmetries along with four standard SUSYs. The appearance of these *nongeneric* SUSYs are closely related to the existence of Killing–Yano tensors [6].

In spite of the complexity of the equations, we are able to present special solutions for the motion on a cone and on a plane. The supersymmetric constraint $Q = 0$ (25) plays a very important role for the forms of solutions.

The results we obtain show spin dependence of the time dilation and of the orbits of the particles in a gravitational field. This leads to the existence of a gravitational analogue of the Stern-Gerlach-type forces well known to appear in electromagnetic phenomena.

Although the Killing tensor $K_{\mu\nu}$ given in (64) defines a constant of motion (directly) for spinless particles in the NUT-Taub space, it requires for spinning particles the nontrivial contributions from spin which involve Killing vector and Killing scalar computed in (65) and (66).

The NUT-Taub space is the Schwarzschild space generalized with NUT or magnetic monopole parameter n. The monopole hypothesis was propounded by Dirac relatively long ago. The ingenious suggestion by Dirac that magnetic monopole does exist in nature was neglected because of the failure to identify such thing. In recent years, however, the development of gauge theories has shed new light on it. The result of this paper is interesting in view of this consideration. With $n = 0$ our result reduces to that of the Schwarzschild space [8].

Supersymmetry/supergravity is relevant in the fundamental theory of particle interactions. It is thus not inconceivable that nature might make some use of it. In regard to this, the study of the geometry of graded pseudo-manifolds with both real number and anticommuting variables is well justified.

Acknowledgements One of the authors (Akhtara Banu) would like to thank the Chairman of the UGC, Bangladesh, for granting fellowship, during this work.

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